

TERMINAL VALUATIONS AND THE NASH PROBLEM

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ABSTRACT. Let X be an algebraic variety of characteristic zero. Terminal valuations are defined in the sense of the minimal model program, as those valuations given by the exceptional divisors on a minimal model over X . We prove that every terminal valuation over X is in the image of the Nash map, and thus it corresponds to a maximal family of arcs through the singular locus of X . In dimension two, this result gives a new proof of the theorem of Fernández de Bobadilla and Pe Pereira stating that, for surfaces, the Nash map is a bijection.

1. INTRODUCTION

Working in characteristic zero, the space of formal arcs passing through the singular points of an algebraic variety X decomposes into finitely many irreducible families, and carries some of the essential information encoded in a resolution of singularities. The *Nash map* associates a divisorial valuation to every maximal irreducible family of arcs through the singular locus of X [Nas95]. In this paper, we will refer to these valuations as the *Nash valuations* over X .

The Nash problem asks for a geometric characterization of Nash valuations in terms of resolutions of X . To this end, Nash introduced the notion of *essential valuations* as those divisorial valuations whose center on every resolution is an irreducible component of the inverse image of the singular locus of X . It is easy to see that every Nash valuation is essential, and Nash asked whether the converse is also true. Regarded as a function to the set of essential valuations, the question is whether the Nash map is surjective.

The Nash problem was successfully settled in dimension two in [FdBPP12] using topological arguments (we refer to their paper for a comprehensive list of references on previous results). In higher dimensions, the characterization of Nash valuations as essential valuations is known to hold for toric singularities and in some other special cases [IK03, Ish05, Ish06, GP07, PPP08, LJR12, LA]. However, examples showing that the Nash map is not always surjective were found in all dimensions ≥ 3 [IK03, dF13, JK13]. In view of these examples, one should rephrase the problem by asking whether there is some other way to characterize the image of the Nash map.

In this paper, we approach this problem from the point of view of the minimal model program. Recall that a minimal model over X is a projective birational morphism $f: Y \rightarrow X$ from a normal variety Y with terminal singularities such that the canonical class K_Y is relatively nef over X . We say that a divisorial valuation ν on X is *terminal with respect to the minimal model program over X* , or simply that it is a *terminal valuation* over X , if there exists a prime exceptional divisor E on a minimal model $f: Y \rightarrow X$ such that $\nu = \text{ord}_E$.

Our main result provides a characterization of a subset of the image of the Nash map as the one consisting of terminal valuations.

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Theorem 1.1. *Every terminal valuation over X is a Nash valuation.*

Since, in dimension two, terminal valuations and essential valuations are clearly the same (both are the valuations defined by the exceptional divisors on the minimal resolution), we obtain a new, purely algebro-geometric proof of the main theorem of [FdBPP12].

Corollary 1.2 ([FdBPP12]). *The Nash map is a bijection in dimension two.*

Theorem 1.1 is the natural generalization of this result to higher dimensions. It implies, for instance, that the Nash map is surjective whenever there exists a nonsingular minimal model over X with exceptional locus of pure codimension one. Since every exceptional divisor over a variety with terminal singularities is uniruled [HM07], Theorem 1.1 also implies, as a by-product, that every divisorial valuation defined by a divisor that is not uniruled is necessarily a Nash valuation. This recovers, in particular, the main result of [LJR12], whose proof is however simpler and more direct.

At a first sight, one may wonder whether all Nash valuations are terminal valuations. While this is the case in dimension two, it fails for simple reasons in higher dimensions. For instance, it is clear that there are no terminal valuations over a variety with terminal singularities, or over a variety which admits a small resolution. On the other hand, there are always Nash valuations over any singular variety.

More examples showing that not all Nash valuations are terminal valuations can be constructed using toric geometry. In §6, we give a toric description of terminal valuations over a toric variety and compare it with the description of Nash valuations given in [IK03].

In conclusion, Theorem 1.1 should be viewed as complementing the fact that Nash valuations are essential. We have inclusions

$$\{\text{terminal valuations}\} \subset \{\text{Nash valuations}\} \subset \{\text{essential valuations}\},$$

but either inclusion can be strict.

1.1. Outline of the proof. We were led to consider minimal models (and define, accordingly, terminal valuations) as a result of our attempt to understand, from an algebro-geometric standpoint, some of the topological computations carried out in [FdBPP12]. The idea of looking at divisors on minimal models in connection to the Nash problem was also suggested by Fernández de Bobadilla.

For simplicity, let us focus on the two-dimensional case, as the main ideas of the proof are already there. So, let X be a surface. Let $f: Y \rightarrow X$ be the minimal resolution, and let $f_\infty: Y_\infty \rightarrow X_\infty$ be the map induced on arc spaces. For every irreducible component E of the exceptional locus $\text{Ex}(f)$, let $E^\circ \subset E$ denote the largest open set disjoint from the other irreducible components of $\text{Ex}(f)$. Then let $N_E^\circ := f_\infty(\pi_Y^{-1}(E^\circ))$ where $\pi_Y: Y_\infty \rightarrow Y$ is the natural projection, and let $N_E \subset X_\infty$ be the closure of N_E° .

Assuming that the Nash problem fails for X , one can find two irreducible components E and F of $\text{Ex}(f)$ such that $N_E \subsetneq N_F$. The idea, due to Lejeune-Jalabert [LJ80], is to detect this by producing a morphism $\Phi: \text{Spec } K[[s]] \rightarrow X_\infty$, for some field extension K/k , such that

$$\Phi(0) \in N_E^\circ \quad \text{and} \quad \Phi(\eta) \in N_F \setminus N_E.$$

Here 0 is the closed point of $\text{Spec } K[[s]]$, and η is its generic point. Such a map is called a *wedge*. The existence of the wedge Φ is a delicate issue, and is established in [Reg06]. Note that Φ does not factor through Y_∞ .

Let us assume that $K = k$. The wedge Φ can be regarded as a morphism

$$\Phi: S = \text{Spec } k[[s, t]] \rightarrow X$$

that does not lift to Y .

At this point, the approach in [FdBPP12] is roughly the following. Reducing to work over \mathbb{C} , and using a suitable approximation theorem, one can assume that Φ is locally given by power series with positive radius of convergence, and thus assume without loss of generality that $S \subset \mathbb{C}^2$ is the product of two open disks of radius 1 [FdB12]. The curves $(s = \lambda) \subset S$, for $0 < |\lambda| < 1$, lift to Y and degenerate, as $\lambda \rightarrow 0$, to a cycle $\sum a_i E_i + T'$ that is supported on the union of the exceptional locus $\text{Ex}(f)$ of f and the image T' of the t -axis $(s = 0) \subset S$. The contradiction is then reached by computing the Euler characteristic of these curves in two ways, as images of small disks, and as they degenerate inside a small tubular neighborhood of $\text{Ex}(f) \cup T'$. The contradiction, resulting from the computation, stems from the fact $\text{Ex}(f)$ does not contain any rational curve with self-intersection -1 .

In order to translate this into algebro-geometric language, we take a resolution of indeterminacy of the rational map $f^{-1} \circ \Phi: S \dashrightarrow Y$. We construct the resolution by taking a minimal sequence of blow-ups of maximal ideals. This gives us a diagram

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & Y \\ g \downarrow & & \downarrow f \\ S & \xrightarrow{\Phi} & X \end{array}$$

where Z is a regular two-dimensional scheme. Then we shift the computation from Y to Z .

This reduction has several advantages. First, it allows us to bypass the use of approximation theorems and to work directly in the formal setting. Furthermore, we avoid having to deal with the singularities of $\text{Ex}(f)$ and can work in fact with partial resolutions $Y \rightarrow X$. Finally, working on Z allows us to extend the computation to all higher dimensions, by taking wedges defined over suitable field extensions K/k .

The proof relies on the analysis of the ramification of the map ϕ at the generic point of the component G of $\text{Ex}(g)$ intersecting the proper transform T of the t -axis $(s = 0) \subset S$.

We consider the contraction $h: Z \rightarrow Z'$ of all the irreducible components of $\text{Ex}(g)$ that are contracted by ϕ . The map ϕ factors through h and a morphism $\phi': Z' \rightarrow Y$. We look at the relative canonical divisor $K_{Z'/Y}$ of ϕ' , which we decompose as $K_{Z'/Y} = K_{Z'/Y}^{\text{exc}} + K_{Z'/Y}^{\text{hor}}$ by separating those components that are exceptional over S from those that are not. By a local computation in codimension one (like in Hürwitz formula), we check that

$$\text{ord}_G(h^* K_{Z'/Y}^{\text{exc}}) < \text{ord}_G(\phi^* E).$$

On the other hand, a negativity lemma (essentially the Hodge index theorem) implies that

$$\text{ord}_G(h^* K_{Z'/Y}^{\text{exc}}) \geq \text{ord}_G(K_{Z/S}).$$

This is the step where we use the assumption that Y is a minimal model. To conclude, we just observe that $\text{ord}_G(K_{Z/S}) \geq 1$ since S is smooth, and $\text{ord}_G(\phi^* E) = 1$ since T maps to an arc on Y with order of contact one along E . This gives the contradiction we wanted.

Remark 1.3. The computation at the core of the proof becomes particularly transparent if $Z' = Z$ (and Y is smooth). In this special case, $K_{Z/Y}$ is the effective divisor defined by the vanishing of the Jacobian of ϕ , and its coefficient at G is zero because otherwise the special arc $\Phi(0)$ of the wedge could not meet transversally E in Y . The Jacobian of ϕ gives a homomorphism from $\mathcal{O}_Z(\phi^* K_Y)$ to $\mathcal{O}_Z(K_Z)$, and hence we have the linear equivalence $K_{Z/Y} \sim K_Z - \phi^* K_Y$. Note, on the other hand, that K_Z is linear equivalent to the exceptional divisor $K_{Z/S}$, which is defined by the Jacobian of g . Combining these linear equivalences, we

obtain the equivalence $K_{Z/S} - K_{Z/Y}^{\text{exc}} \sim \phi^* K_Y + K_{Z/Y}^{\text{hor}}$. The divisor in the right hand side is g -nef because $K_{Z/Y}^{\text{hor}}$ is horizontal and effective and K_Y is f -nef, and the divisor in the left hand side is g -exceptional. It follows by the negative definiteness of the intersection form of the exceptional divisors of Z that $K_{Z/S} - K_{Z/Y}^{\text{exc}}$ is anti-effective. This is impossible, however, since the coefficient of G in $K_{Z/S}$ is positive and its coefficient in $K_{Z/Y}$ is zero.

Remark 1.4. It would be interesting to try to extend the proof to positive characteristics. There are some difficulties to overcome. First of all, the proof of Theorem 1.1 relies on Reguera's curve selection lemma which requires, in the construction of the wedge, to take a field extension which may in principle be inseparable. This creates problems for the definition of $K_{Z'/Y}$. Notice that this issue does not occur in dimension two, since in this case we are led to work with wedges defined over finite algebraic extensions of k , and if k is algebraically closed then there is only the trivial one. However, one runs into another difficulty: the local computation leading to the first inequality displayed above breaks down if ϕ' is wildly ramified. In order to extend the result of this paper to positive characteristics, it seems that one would need to gain control on the construction of the wedge to avoid these problems.

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2. NOTATION AND CONVENTIONS

We work over an uncountable algebraically closed field k of characteristic zero. All schemes are separated and defined over k . A *point* of a scheme is a schematic point; more generally, we will also consider A -valued points on schemes, where A is any k -algebra. We denote by κ_p the residue field of a point p . A point of an irreducible scheme X is *very general* if it can be chosen arbitrarily in the complement of a countable union of proper closed subsets. If X is a variety, we denote by X_{sm} the smooth locus of X , and set $X_{\text{sing}} = X \setminus X_{\text{sm}}$. The *exceptional locus* of a proper birational morphism of schemes $f: Y \rightarrow X$ is the smallest set $\text{Ex}(f) \subset Y$ away from which f is an isomorphism to its image. A *divisorial valuation* on a variety X is a valuation of the form $\nu = \text{ord}_E$ where E is a prime divisor on a resolution of singularities $Y \rightarrow X$.

3. THE NASH PROBLEM

In this section, we quickly recall basic definitions and properties related to arc spaces and the Nash map. For a full treatment on arc spaces, we refer the reader to [DL99, EM09]. For a more comprehensive introduction to the Nash problem, we refer to [Nas95, IK03].

3.1. Arc space. For any scheme X over k and any field extension K/k , a K -valued arc of X is a morphism $\gamma: \text{Spec } K[[t]] \rightarrow X$. We denote the image of the closed point by $\gamma(0)$ and the image of the generic point by $\gamma(\eta)$.

Let now X be a scheme of finite type over k . For every $m \in \mathbb{N}$, the functor that to any k -algebra A associates $\text{Hom}_{k\text{-Sch}}(\text{Spec } A[t]/(t^{m+1}), X)$ is representable by a scheme of finite type over k , denoted X_m and called the m -th jet scheme of X . For $m > n$, the natural quotients $A[t]/(t^{m+1}) \rightarrow A[t]/(t^{n+1})$ induce morphisms $X_m \rightarrow X_n$, known as the *truncation maps*. The truncation maps are affine, and can be used to define a scheme $X_\infty = \varprojlim X_m$,

called the *arc space* of X . The set of K -valued points of X_∞ is

$$X_\infty(K) = \text{Hom}_{k\text{-Sch}}(\text{Spec } K[[t]], X),$$

the set of K -valued arcs of X . The Zariski topology of X_∞ coincides with the inverse limit topology.

The arc space comes equipped with natural projections $X_\infty \rightarrow X_m$. When $m = 0$, this gives a morphism $\pi_X: X_\infty \rightarrow X$, mapping an arc $\gamma \in X_\infty$ to $\gamma(0) \in X$. If γ is any K -valued arc, then we regard $\pi_X(\gamma)$ as the K -valued point of X given by pre-composition with the inclusion $\text{Spec } K \hookrightarrow \text{Spec } K[[t]]$.

Given a morphism $f: Y \rightarrow X$ of k -schemes of finite type, we obtain an induced morphism between the respective arc spaces $f_\infty: Y_\infty \rightarrow X_\infty$. At the level of the functors of points, f_∞ is just given by composition with f , mapping a K -valued arc $\gamma: \text{Spec } K[[t]] \rightarrow Y$ to the K -valued arc $f \circ \gamma: \text{Spec } K[[t]] \rightarrow X$.

Every K -valued arc $\alpha: \text{Spec } K[[t]] \rightarrow X$ defines a valuation on the local ring $\mathcal{O}_{X, \pi_X(\alpha)}$, given by $\text{ord}_\alpha(h) := \text{ord}_t(\alpha^\sharp(h))$. The valuation extends to a valuation of the function field of X if and only if $K((t))$ is a field extension of the function field of X , that is, if and only if the image of α is dense in X .

3.2. Nash valuations and the Nash map. A divisorial valuation ν is an *essential valuation* over X if the center of ν on any resolution of X is an irreducible component of the inverse image of X_{sing} , and is a *Nash valuation* over X if it is the valuation defined by the generic point of an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$.

The decomposition of $\pi_X^{-1}(X_{\text{sing}})$ into irreducible components, and the way these components define divisorial valuations over X , can be understood using resolutions of singularities. This is well explained, for instance, in [IK03]. We briefly outline the argument.

Let $f: Y \rightarrow X$ be any resolution of singularities, and let

$$f^{-1}(X_{\text{sing}}) = \bigcup_{i \in I} C_i$$

be the decomposition of the inverse image of the singular locus into irreducible components. Notice that I is a finite set. Let $Y_i \rightarrow Y$ be the normalized blow-up of C_i , and let E_i be the prime divisor on Y_i dominating C_i .

Since Y is smooth, all truncation maps $Y_m \rightarrow Y_n$ are smooth, and this implies that each $\pi_Y^{-1}(C_i) \subset Y_\infty$ is an irreducible set. Then, denoting by N_i the closure of $f_\infty(\pi_Y^{-1}(C_i))$ in X_∞ , each N_i is irreducible, and there is a unique minimal subset $I' \subset I$ such that

$$\pi_X^{-1}(X_{\text{sing}}) = \bigcup_{i \in I'} N_i.$$

For every $i \in I'$, the generic point α_i of N_i is the image of the generic point of $\pi_{Y_i}^{-1}(E_i)$, and this implies that $\text{ord}_{\alpha_i} = \text{ord}_{E_i}$. In particular, each valuation ord_{α_i} is a divisorial valuation on X .

Given the above resolution f , the center in Y of any essential valuation over X must be one of the irreducible components C_i . This means that there is a subset $I'' \subset I$ such that a divisorial valuation ν over X is essential if and only if $\nu = \text{ord}_{E_i}$ for some $i \in I''$. The decomposition of $\pi_X^{-1}(X_{\text{sing}})$ into irreducible components only depends on the topology of X_∞ , and not on the choice of resolution. This implies that $I' \subset I''$. In terms of valuations, it means that there is an inclusion

$$\{\text{Nash valuations}\} \hookrightarrow \{\text{essential valuations}\}.$$

This is the *Nash map*. The original formulation of the *Nash problem* asked whether this map is surjective.

3.3. Wedges. Let X be a scheme of finite type over k . For any field extension K/k , a (parametrized) K -valued *wedge* on X is a morphism

$$\Phi: \operatorname{Spec} K[[s, t]] \rightarrow X.$$

Since $K[[s, t]] = K[[s]][[t]]$, giving a K -valued wedge $\Phi: \operatorname{Spec} K[[s, t]] \rightarrow X$ is equivalent to giving a $K[[s]]$ -valued arc (which we still denote by the same letter)

$$\Phi: \operatorname{Spec} K[[s]] \rightarrow X_\infty.$$

Restricting to the closed and generic points of $\operatorname{Spec} K[[s]]$, we obtain arcs

$$\Phi_0: \operatorname{Spec} K \rightarrow X_\infty \quad \text{and} \quad \Phi_\eta: \operatorname{Spec} K((s)) \rightarrow X_\infty.$$

We can think of Φ_0 as being a specialization of Φ_η . We call Φ_0 the *special arc* of the wedge, and Φ_η the *generic arc* of the wedge.

Remark 3.1. It is important to note that in this discussion we are using the explicit choice of coordinates (s, t) on $\operatorname{Spec} K[[s, t]]$. When talking about wedges, we implicitly assume that such a choice of coordinates has been made.

Let $f: Y \rightarrow X$ be a projective birational morphism of algebraic varieties over k , and let

$$\Phi: S = \operatorname{Spec} K[[s, t]] \rightarrow X$$

be a wedge. We say that Φ *lifts* to Y if it factors through f . Thinking of Φ as an arc on X_∞ , we also say that Φ lifts to Y_∞ .

If both the special arc Φ_0 and the generic arc Φ_η are not contained in the indeterminacies of f^{-1} , then, by the valuative criterion of properness, both arcs lift (uniquely) to arcs on Y :

$$\begin{array}{ccc} & Y & \\ \tilde{\Phi}_0 \nearrow & \downarrow f & \nwarrow \tilde{\Phi}_\eta \\ \operatorname{Spec} K[[t]] & \xrightarrow{\Phi_0} X, & \operatorname{Spec} K((s))[[t]] \xrightarrow{\Phi_\eta} X. \end{array}$$

However, in general $\tilde{\Phi}_0$ will not be a specialization of $\tilde{\Phi}_\eta$. In fact, $\tilde{\Phi}_\eta$, viewed as a $K((s))$ -valued point on Y_∞ , might fail to extend to an arc $\operatorname{Spec} K[[s]] \rightarrow Y_\infty$. This is related to whether Φ lifts to Y or not. The following result follows immediately from the definitions.

Lemma 3.2. *With above terminology, the following are equivalent:*

- (a) Φ lifts to Y ,
- (b) the morphism $\tilde{\Phi}_\eta: \operatorname{Spec} K((s))[[t]] \rightarrow Y$ factors through $\operatorname{Spec} K[[s, t]]$,
- (c) the morphism $\tilde{\Phi}_\eta: \operatorname{Spec} K((s)) \rightarrow Y_\infty$ extends to an arc $\tilde{\Phi}_\eta: \operatorname{Spec} K[[s]] \rightarrow Y_\infty$.

3.4. The curve selection lemma. The idea outlined in §1.1 to look at the Nash problem as a lifting problem for wedges was first considered in [LJ80], where the notion of wedges was introduced in the context of surface singularities. This approach is not specific of dimension two, and provides a natural way to address the Nash problem in all dimensions.

Given two closed irreducible sets M and N in X_∞ , with $N \subsetneq M$, the existence of a morphism $\Phi: \operatorname{Spec} K[[s]] \rightarrow X_\infty$ such that $\Phi(0) \in N$ and $\Phi(\eta) \in M \setminus N$ should be viewed as an algebro-geometric analogue of Milnor's curve selection lemma.

In general, the curve selection lemma holds, in the algebro-geometric sense, for Noetherian schemes, where it can be proven by cutting down and using induction on dimension. However,

X_∞ is not a Noetherian scheme, and there are examples where the curve selection lemma fails in the non-Noetherian setting (e.g., see [FdBPP, Example 4]). It is therefore a delicate issue to establish the existence of a wedge Φ with the above properties.

The first general result on the curve selection lemma for arc spaces is due to Reguera [Reg06]. In the proof of Theorem 1.1 we will use the following application of Reguera's result. We first introduce some notation.

Let $f: Y \rightarrow X$ be a projective birational morphism from a normal variety Y , and let E be a prime divisor on Y contained in $f^{-1}(X_{\text{sing}})$. Let $\tilde{\alpha} \in Y_\infty$ be the generic point of the irreducible component of $\pi_Y^{-1}(E)$ dominating E , and let $\alpha = f_\infty(\tilde{\alpha})$. Note that $\alpha \in \pi_X^{-1}(X_{\text{sing}})$. The closure of α is what was denoted by N_E in §1.1. Recall that a point of a scheme is said to be very general if it is in the complement of a countable union of preassigned proper closed subsets.

Theorem 3.3. *With the above notation, suppose that α is not the generic point of an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$. Then for a very general point $p \in E$ with residue field κ_p , there is a finite algebraic extension K/κ_p , and a wedge*

$$\Phi: K[[s, t]] \rightarrow X$$

that does not lift to Y , such that the lift $\tilde{\Phi}_0: \text{Spec } K[[t]] \rightarrow Y$ of the special arc Φ_0 has order of contact one with E at p .

This result follows from [Reg06, Corollary 4.6] by a specialization argument along the lines of [LJR12, §2.2] or [FdB12, §3.2]. We explain the argument in details in the Appendix.

4. GEOMETRY ON A RESOLUTION OF A WEDGE

Throughout this section, let $f: Y \rightarrow X$ be a projective birational morphism of varieties over k , and consider a wedge

$$\Phi: S = \text{Spec } K[[s, t]] \rightarrow X$$

where K/k is a field extension. We assume that Φ does not lift to Y , and that the image of Φ in X is not fully contained in the indeterminacy locus of f^{-1} .

4.1. Resolution of the wedge. We can identify f with the blow-up of an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ that does not vanish on the image of Φ . Consider then the inverse image $\mathcal{J} = \Phi^{-1}\mathcal{I} \cdot \mathcal{O}_S$. We write $\mathcal{J} = \mathfrak{a} \cdot (h)$ where $h \in K[[s, t]]$ and \mathfrak{a} is an (s, t) -primary ideal.

Proposition 4.1. *There is a commutative diagram*

$$\begin{array}{ccccc} Z & & & & \\ & \searrow h & & \searrow \phi & \\ & & Z' & \xrightarrow{\phi'} & Y \\ & \searrow g & \downarrow g' & & \downarrow f \\ & & S & \xrightarrow{\Phi} & X \end{array}$$

where g' is the blow-up of the integral closure $\bar{\mathfrak{a}}$ of \mathfrak{a} , and g is a minimal sequence of blow-ups of maximal ideals such that h is a morphism. Both Z and Z' are two-dimensional and projective over S , Z is regular, and Z' is normal and \mathbb{Q} -factorial. The exceptional loci $\text{Ex}(g)$ and $\text{Ex}(g')$ are non-empty sets of pure codimension one, and their irreducible components are projective curves.

Proof. The morphism ϕ' exists by the universal property of the blow-up.

Being an (s, t) -primary ideal, $\bar{\alpha}$ is generated by polynomials, that is, $\bar{\alpha} = \bar{\alpha}^\circ \cdot K[[s, t]]$ for an (s, t) -primary ideal $\bar{\alpha}^\circ \subset K[s, t]$. Note that $\bar{\alpha}^\circ = \bar{\alpha} \cap K[s, t]$ is integrally closed. If $S^\circ = \text{Spec } K[s, t]$ and $(Z')^\circ \rightarrow S^\circ$ is the blow-up of $\bar{\alpha}^\circ$, then we have that $Z' = (Z')^\circ \times_{S^\circ} S$. In particular, Z' is normal and projective over S .

The indeterminacies of $S^\circ \dashrightarrow (Z')^\circ$ can be resolved by a minimal sequence $Z^\circ \rightarrow S^\circ$ of blow-ups of maximal ideals. All centers of blow-up are closed points, and thus they lie over the closed point of S° . We let $Z = Z^\circ \times_{S^\circ} S$. By base change, we obtain a morphism $g: Z \rightarrow S$, resolving the indeterminacies of $S \dashrightarrow Z'$, given by a composition of blow-ups of maximal ideals. We let h and ϕ be the induced morphisms.

To check that Z' is \mathbb{Q} -factorial, we apply a result of Lipman. Since S has rational singularities, we have $H^1(Z, \mathcal{O}_Z) = 0$ by [Lip69, Proposition (1.2)]. Using [Lip69, Lemma (12.2)], we see that this implies that $H^1(h^{-1}(U), \mathcal{O}_Z) = 0$ for every open set $U \subset Z'$, and therefore Z' has rational singularities. Then [Lip69, Proposition (17.1)] implies that Z' is \mathbb{Q} -factorial.

By construction, all irreducible components of $\text{Ex}(g)$ and $\text{Ex}(g')$ are projective. These sets are non-empty because $f^{-1} \circ \Phi: S \dashrightarrow Y$ is not a morphism. \square

We say that an irreducible component C of $\text{Ex}(g)$ is *contracted* by ϕ if ϕ maps every point of C to the same point of Y . A similar definition is given for ϕ' and the components of $\text{Ex}(g')$.

Lemma 4.2. *For an irreducible component C of $\text{Ex}(g)$, the following are equivalent:*

- (a) *C is a component of $\text{Ex}(h)$;*
- (b) *C is contracted by ϕ ;*
- (c) *there is a closed point $q \in C$ such that if $c \in C$ is the generic point, then $\phi(q) = \phi(c)$.*

Proof. It is clear from the definition that $(a) \Rightarrow (b) \Rightarrow (c)$. Suppose then that C is not a component of $\text{Ex}(h)$. Then $h(C)$ is a component of $\text{Ex}(g')$. We can assume that X is affine. The line bundle $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ on Y is relatively very ample over X . By construction, Z' is the normalization of the blow-up of $\Phi^{-1}\mathcal{I} \cdot \mathcal{O}_S$, and therefore the line bundle $(\phi')^*\mathcal{L}$ is relatively ample over S . Note that $(\phi')^*\mathcal{L}$ is globally generated by sections that are pulled back from sections of \mathcal{L} . In particular, for every closed point $q \in C$ we can find a section $s \in H^0(Y, \mathcal{L})$ such that $(\phi^*s)(q) = 0$ and $(\phi^*s)(c) \neq 0$. This means that $s(\phi(p)) = 0$ and $s(\phi(c)) \neq 0$, and hence $\phi(q) \neq \phi(p)$. We conclude that $(c) \Rightarrow (a)$. \square

Since Z is smooth, every divisor on Z is Cartier. Note that every divisor on Z is linearly equivalent to a unique g -exceptional divisor. The intersection product of a divisor D with an irreducible component C of $\text{Ex}(g)$, is defined by $D \cdot C := \deg(\mathcal{O}_Z(D)|_C)$ (cf. [Lip69, §10]). This product extends to all \mathbb{Q} -divisors on Z . A \mathbb{Q} -divisor D is g -nef (resp., h -nef) if $D \cdot C \geq 0$ for every irreducible component C of $\text{Ex}(g)$ (resp., of $\text{Ex}(h)$). Note that if D is effective and its support does not contain any irreducible component of $\text{Ex}(g)$, then D is g -nef.

If C_1, \dots, C_r are the irreducible components of $\text{Ex}(g)$, then the intersection matrix $(C_i \cdot C_j)$ is negative definite (e.g., see [Lip69, Lemma (14.1)]). One deduces the following negativity lemma (cf. [KM98, Lemma 3.41]).

Lemma 4.3. *For any subset $I \subset \{1, \dots, r\}$, let $D = \sum_{i \in I} a_i C_i$ be a \mathbb{Q} -divisor such that $D \cdot C_i \geq 0$ for every $i \in I$. Then $a_i \leq 0$ for all i .*

Let L be a \mathbb{Q} -Cartier \mathbb{Q} -divisor on Y , and assume that $\phi(Z)$ is not contained in the support of L . Let r be a positive integer such that rL is Cartier. By pulling back the local equations of rL to Z and dividing by r , we define the pull-back ϕ^*L , which is a \mathbb{Q} -divisor on Z .

Lemma 4.4. *With the above notation, if L is f -nef then ϕ^*L is g -nef.*

Proof. We can assume without loss of generality that X is affine. Let C be any irreducible component of $\text{Ex}(g)$. By the continuity of the degree function on \mathbb{Q} -divisors on C , it suffices to prove that $\phi^*L \cdot C \geq 0$ when L is ample. But this is clear, since in this case $\mathcal{O}_Y(mL)$ is globally generated for some $m \geq 1$, and thus we can write $L \sim \frac{1}{m}H$ where H is an effective divisor that does not contain the image of C . \square

Since Z' is \mathbb{Q} -factorial, we can define the intersection product $D \cdot C$ between a divisor D on Z' and an irreducible component C of $\text{Ex}(g')$. As usual, we say that D is g' -nef if $D \cdot C = 0$ for every C . The pull-back h^*D is also defined, by rescaling. Notice that D is g' -nef if and only if h^*D is g -nef.

4.2. Canonical divisors. The next step is to define a canonical divisor of Z . This is not straightforward, as Z is not a variety over a field. Since Z is a scheme of finite type over $K[[s, t]]$, we need to work with *special differentials*. We follow the treatment given in [dFEM11, Appendix A].

Let $R = F[[x_1, \dots, x_n]]$ where F is a field of characteristic zero. For any R -module M , an F -derivation $D: R \rightarrow M$ is said to be a *special F -derivation* if $D(h) = \sum_i \frac{\partial h}{\partial x_i} D(x_i)$ for every $h \in R$. If A is an R -algebra and M is an A -module, then the module $\text{Der}'_F(A, M)$ of *special F -derivations* consists of all derivations $D: A \rightarrow M$ that restrict to special F -derivations on R . For any R -algebra A there is an A -module $\Omega'_{A/F}$ and a special F -derivation $d'_{A/F}: A \rightarrow \Omega'_{A/F}$ that induces an isomorphism of A -modules

$$\text{Hom}_A(\Omega'_{A/F}, M) \rightarrow \text{Der}'_F(A, M)$$

for every A -module M [dFEM11, Corollary A.4]. The module $\Omega'_{A/F}$ is called the *module of special differentials* of A over F . As for usual differentials, special differentials commute with localization [dFEM11, Lemma A.7]. It follows then by the second part of [dFEM11, Corollary A.4] that if A is essentially of finite type over R , then $\Omega'_{A/F}$ is a finitely generated A -module.

For any scheme T over R , we obtain a coherent sheaf $\Omega'_{T/F}$, called the sheaf of *special differentials* of T over F . For example, if $T = \text{Spec } R$, then $\Omega'_{T/F}$ is the free \mathcal{O}_T -module generated by dx_1, \dots, dx_n [dFEM11, Lemma A.2]. If T is essentially of finite type over R , then $\Omega'_{T/F}$ is a coherent sheaf. If furthermore T is smooth of pure dimension a , and the residue field κ_p of a closed point $p \in T$ has transcendence degree b over F , then it follows from [dFEM11, Proposition A.8] that $\Omega'_{T/F}$ is a locally free sheaf of rank $a + b$.

Suppose now that T is a reduced scheme essentially of finite type over R , and that F is a finitely generated extension of a subfield E . Note that T is a scheme essentially of finite type over $E[[x_1, \dots, x_n]]$, too. If d is the transcendence degree of F/E , then $\Omega_{F/E}$ is an F -vector space of dimension d .

Proposition 4.5. *With the above assumptions, there is a short exact sequence*

$$0 \rightarrow \Omega_{F/E} \otimes \mathcal{O}_T \rightarrow \Omega'_{T/E} \rightarrow \Omega'_{T/F} \rightarrow 0.$$

Proof. For every affine chart $\text{Spec } A \subset T$, and every A -module M , we have an exact sequence

$$0 \rightarrow \text{Der}'_F(A, M) \rightarrow \text{Der}'_E(A, M) \rightarrow \text{Der}_E(F, M) \otimes A.$$

This implies that the sequence in the statement is exact in the middle and on the right. The exactness on the left follows by observing that $\Omega_{F/E} \otimes \mathcal{O}_T \cong \mathcal{O}_T^{\otimes d}$, and the difference between the ranks of $\Omega'_{T/E}$ and $\Omega'_{T/F}$ is equal to d . \square

We now come back to our setting. Since Z is regular of dimension two and the residue fields of its closed points are finite extensions of K , $\Omega'_{Z/K}$ is a locally free sheaf, of rank two. Then there is a divisor K_Z on Z such that

$$\mathcal{O}_Z(K_Z) \cong \wedge^2 \Omega'_{Z/K}.$$

We say that K_Z is a *canonical divisor* of Z .

The natural map of invertible sheaves

$$\wedge^2 \Omega'_{S/K} \otimes \mathcal{O}_Z \rightarrow \wedge^2 \Omega'_{Z/K}$$

is defined by multiplication of a local equation of an effective divisor supported on the exceptional locus $\text{Ex}(g)$ [dFEM11, Lemma A.11]. We denote this divisor by $K_{Z/S}$ and call it the *relative canonical divisor* of Z over S . Note that, since $\wedge^2 \Omega'_{S/K} \cong \mathcal{O}_S$, we have

$$(4.1) \quad K_Z \sim K_{Z/S}.$$

If K/k is a finitely generated field extension, then Z can also be regarded as a scheme essentially of finite type over $k[[s, t]]$. The sheaf $\Omega'_{Z/k}$ of special differentials of Z over k is a locally free sheaf of rank $d + 2$, where d is the transcendence degree of K/k . It follows by Proposition 4.5 that

$$\mathcal{O}_Z(K_Z) \cong \wedge^{d+2} \Omega'_{Z/k}.$$

In particular, the definition of canonical divisor on Z is independent of whether we take (special) differentials over K or over k .

Remark 4.6. With the notation as in the proof of Proposition 4.1, it follows by [dFEM11, Proposition A.10] that $\Omega_{Z^\circ/K} \otimes \mathcal{O}_Z \cong \Omega'_{Z/K}$ (and, similarly, $\Omega_{Z^\circ/k} \otimes \mathcal{O}_Z \cong \Omega'_{Z/k}$). In particular, $K_{Z/S}$ is the pull-back of K_{Z°/S° , and K_Z is linearly equivalent to the pull-back of K_{Z° .

4.3. Relative canonical divisors. For the reminder of this section, we let $n = \dim Y$. We assume henceforth that K/k has transcendence degree $d = n - 2$, and ϕ is dominant.

Let $a: \tilde{Y} \rightarrow Y$ be a projective resolution of singularities. By taking a sequence of blow-ups $b: \tilde{Z} \rightarrow Z$ centered at closed points, we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\phi}} & \tilde{Y} \\ b \downarrow & & \downarrow a \\ Z & \xrightarrow{\phi} & Y \end{array}$$

where \tilde{Z} is smooth. Note that $\tilde{\phi}$ is dominant.

Let $K_{\tilde{Z}}$ be a canonical divisor of \tilde{Z} , defined by the condition $\mathcal{O}_{\tilde{Z}}(K_{\tilde{Z}}) \cong \wedge^n \Omega'_{\tilde{Z}/k}$. It is important here to keep in mind that we work with divisors, and not divisor classes. By [dFEM11, Lemma A.11], we can choose $K_{\tilde{Z}}$ such that $b_* K_{\tilde{Z}} = K_Z$. We fix a canonical divisor $K_{\tilde{Y}}$.

Consider the natural map

$$\alpha: \Omega_{\tilde{Y}/k} \otimes \mathcal{O}_{\tilde{Z}} \rightarrow \Omega'_{\tilde{Z}/k}.$$

Note that this is a map of locally free sheaves of rank n .

Lemma 4.7. *The map of invertible sheaves*

$$\wedge^n \alpha: \wedge^n \Omega_{\tilde{Y}/k} \otimes \mathcal{O}_{\tilde{Z}} \rightarrow \wedge^n \Omega'_{\tilde{Z}/k}$$

is locally given by multiplication of an equation of an effective divisor $K_{\tilde{Z}/\tilde{Y}}$ linearly equivalent to $K_{\tilde{Z}} - \tilde{\phi}^ K_{\tilde{Y}}$.*

Proof. Let $y \in \tilde{Y}$ and $z \in \tilde{Z}$ be the generic points, and denote by κ_y and κ_z their residue fields. Note that $\tilde{\phi}(z) = y$, since $\tilde{\phi}$ is dominant. Localizing at the generic points, we see that $(\wedge^n \Omega_{\tilde{Y}/k})_y = \kappa_y$ and $(\wedge^n \Omega'_{\tilde{Z}/k})_z = \kappa_z$, and the induced map $(\wedge^n \Omega_{\tilde{Y}/k})_y \otimes \kappa_z \rightarrow (\wedge^n \Omega'_{\tilde{Z}/k})_z$ is an isomorphism. This implies that $\wedge^n \alpha$ is injective, and the assertion follows. \square

Suppose now that Y is a normal variety with a \mathbb{Q} -Cartier canonical divisor K_Y . We can take $K_Y = a_* K_{\tilde{Y}}$. Let $K_{\tilde{Y}/Y} := K_{\tilde{Y}} - a^* K_Y$ be the relative canonical divisor of $a: \tilde{Y} \rightarrow Y$. Then we define the *relative canonical divisor* of ϕ to be

$$K_{Z/Y} := b_*(K_{\tilde{Z}/\tilde{Y}} + \tilde{\phi}^* K_{\tilde{Y}/Y}).$$

Lemma 4.8. *The definition of $K_{Z/Y}$ is independent of the choice of the models \tilde{Y} and \tilde{Z} .*

Proof. It suffices to check that the definition does not change if we replace \tilde{Y} and \tilde{Z} by some other models \hat{Y} and \hat{Z} dominating them. Let $\beta: \hat{Z} \rightarrow \tilde{Z}$ and $\hat{\phi}: \hat{Z} \rightarrow \hat{Y}$ denote the corresponding morphisms. Applying several times the chain rule, we see that

$$K_{\hat{Z}/\hat{Y}} + \hat{\phi}^* K_{\hat{Y}/Y} = K_{\hat{Z}/\tilde{Z}} + \beta^*(K_{\tilde{Z}/\tilde{Y}} + \tilde{\phi}^* K_{\tilde{Y}/Y}).$$

The push-forward of this divisor to \tilde{Z} agrees with $K_{\tilde{Z}/\tilde{Y}} + \tilde{\phi}^* K_{\tilde{Y}/Y}$ by the projection formula and the fact that $K_{\hat{Z}/\tilde{Z}}$ is β -exceptional. \square

Proposition 4.9. *We have*

$$K_{Z/Y} \sim K_Z - \phi^* K_Y.$$

Moreover, if Y has canonical singularities, then $K_{Z/Y} \geq 0$.

Proof. We have

$$K_{\tilde{Z}/\tilde{Y}} + \tilde{\phi}^* K_{\tilde{Y}/Y} \sim K_{\tilde{Z}} - \tilde{\phi}^*(a^* K_Y)$$

by Lemma 4.7. Note that $b_* K_{\tilde{Z}} = K_Z$ and $b_*(\tilde{\phi}^*(a^* K_Y)) = \phi^* K_Y$. Then the first assertion is a consequence of the fact that pushing forward preserves linear equivalence. Regarding the last assertion, it suffices to recall that $K_{\tilde{Z}/\tilde{Y}} \geq 0$, and observe that $K_{\tilde{Y}/Y} \geq 0$ (and hence $\tilde{\phi}^* K_{\tilde{Y}/Y} \geq 0$) if Y has canonical singularities. \square

Proposition 4.10. *Let E be a prime \mathbb{Q} -Cartier divisor on Y , and suppose that C is a prime divisor on Z dominating E . Then*

$$\text{ord}_C(K_{Z/Y}) = \text{ord}_C(\phi^* E) - 1.$$

Proof. Let $p \in Y$ and $q \in Z$ be, respectively, the generic points of E and C , and let κ_p and κ_q be their residue fields. By Cohen structure theorem, the completed local rings $\widehat{\mathcal{O}_{Y,p}}$ and $\widehat{\mathcal{O}_{Z,q}}$ have coefficient fields. We consider the morphism

$$\psi: W = \text{Spec } \widehat{\mathcal{O}_{Z,q}} \longrightarrow V = \text{Spec } \widehat{\mathcal{O}_{Y,p}}$$

induced by ϕ . By localizing at p and applying [dFEM11, Proposition A.10], we see that $\Omega_{Y/k} \otimes \mathcal{O}_V \cong \Omega'_{V/k}$. Similarly, recalling that $\Omega'_{Z/k} \cong \Omega_{Z^\circ/k} \otimes \mathcal{O}_Z$ (cf. Remark 4.6), we see

that $\Omega'_{Z/k} \otimes \mathcal{O}_W \cong \Omega'_{W/k}$ by applying the same argument to the map $W \rightarrow Z^\circ$. Therefore the local equation of $K_{Z/Y}$ in \mathcal{O}_W is given by the determinant of the Jacobian matrix defining the map $\Omega'_{V/k} \otimes \mathcal{O}_W \rightarrow \Omega'_{W/k}$. We have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{\kappa_p/k} \otimes \mathcal{O}_W & \longrightarrow & \Omega'_{V/k} \otimes \mathcal{O}_W & \longrightarrow & \Omega'_{V/\kappa_p} \otimes \mathcal{O}_W \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{\kappa_p/k} \otimes \mathcal{O}_W & \longrightarrow & \Omega'_{W/k} & \longrightarrow & \Omega'_{W/\kappa_p} \longrightarrow 0, \end{array}$$

where the rows are exact by Proposition 4.5 and the fact that Ω'_{V/κ_p} is locally free. From the diagram, we see that $K_{Z/Y}$ is locally defined by the equation of the map $\Omega'_{V/\kappa_p} \otimes \mathcal{O}_W \rightarrow \Omega'_{W/\kappa_p}$. Since the extension κ_q/κ_p is finite, we have $\Omega_{\kappa_q/\kappa_p} \otimes \mathcal{O}_W \cong \mathcal{O}_W$, and therefore $\Omega'_{W/\kappa_p} \cong \Omega'_{W/\kappa_q}$ by Proposition 4.5. Then the assertion follows by the following standard computation. We fix isomorphisms $\mathcal{O}_V \cong \kappa_p[[v]]$ and $\mathcal{O}_W \cong \kappa_q[[w]]$, so that ψ is given by the equation $v = uw^a$ where u is a unit in \mathcal{O}_W and $a = \text{ord}_C(\phi^*E)$. Then Ω'_{V/κ_p} is generated by dv , Ω'_{W/κ_q} is generated by dw , and we have $dv = uaw^{a-1}dw + w^adu$, which shows that $\text{ord}_C(K_{Z/Y}) = a - 1$. \square

Still in the setting of Proposition 4.1, we define the canonical divisor $K_{Z'}$ of Z' and the relative canonical divisor $K_{Z'/Y}$ of ϕ' to be, respectively, h_*K_Z and $h_*K_{Z/Y}$.

The next corollary follows immediately from Proposition 4.9.

Corollary 4.11. *We have $K_{Z'/Y} \sim K_{Z'} - \phi'^*K_Y$. Moreover, if Y has canonical singularities, then $K_{Z'/Y} \geq 0$.*

The *relative canonical divisor* of h is defined by

$$K_{Z/Z'} := K_Z - h^*K_{Z'}.$$

Note that since $K_{Z'} = h_*K_Z$, $K_{Z/Z'}$ is h -exceptional.

Proposition 4.12. *We have $K_{Z/Z'} \leq 0$.*

Proof. By Lemma 4.3, it suffices to show that $K_{Z/Z'}$ is h -nef. This property is well-known to hold, under similar assumptions, for surfaces of finite type over an algebraically closed field. We reduce to that case as follows (we keep the notation as in the proof of Proposition 4.1).

Let \overline{K} be the algebraic closure of K , and let G be the Galois group. The surface $Z_{\overline{K}}^\circ$ is obtained from $S_{\overline{K}}^\circ$ by a minimal sequence of blow-ups of G -orbits of smooth closed points such that the induced rational map $h_{\overline{K}}^\circ: Z_{\overline{K}}^\circ \dashrightarrow (Z')_{\overline{K}}^\circ$ is a morphism. Any irreducible component C of $\text{Ex}(g)$ is the pull-back of a curve C° to Z° , whose pull-back to $Z_{\overline{K}}^\circ$ is a union of finitely many disjoint curves D_1, \dots, D_m forming an orbit under the action of G . Moreover, $K_{Z_{\overline{K}}^\circ}$ and K_Z are the pull-backs (under the respective maps) of K_{Z° (cf. Remark 4.6). If C is h -exceptional, then each D_i is $(h_{\overline{K}}^\circ)$ -exceptional. Therefore we have $K_{Z_{\overline{K}}^\circ} \cdot D_i \geq 0$ by the minimality of the sequence of G -equivariant blow-ups, since otherwise the adjunction formula would imply that D_i is a (-1) -curve. This implies that $K_Z \cdot C \geq 0$, and hence $K_{Z/Z'} \cdot C \geq 0$. \square

5. PROOF OF THEOREM 1.1

Let X be a variety over k . Theorem 1.1 is well-known to hold in dimension one (cf. Remark A.2). We can then assume that $\dim X \geq 2$.

Suppose that ν is a terminal valuation over X , as defined in the introduction. This means that there is a minimal model $f: Y \rightarrow X$, and a prime exceptional divisor E on Y , such that $\nu = \text{ord}_E$. Note that $E \subset f^{-1}(X_{\text{sing}})$. We can assume without loss of generality that Y is \mathbb{Q} -factorial (cf. §6.1). Let $\tilde{\alpha} \in Y_{\infty}$ be the generic point of the irreducible component of $\pi_Y^{-1}(E)$ dominating E , and let $\alpha = f_{\infty}(\tilde{\alpha}) \in X_{\infty}$. Then $\alpha \in \pi_X^{-1}(X_{\text{sing}})$ and $\nu = \text{ord}_{\alpha}$.

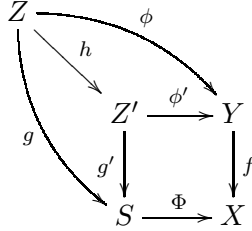
The valuation ν is a Nash valuation over X if and only if α is the generic point of an irreducible component of $\pi_X^{-1}(X_{\text{sing}})$. We suppose by way of contradiction that this is not the case. Then we are in the setting of Theorem 3.3.

Let $p \in E$ be a very general point of codimension one. By Theorem 3.3, there is a finite algebraic extension K/κ_p , and a wedge

$$\Phi: \text{Spec } K[[s, t]] \rightarrow X$$

that does not lift to Y , such that the lift $\tilde{\Phi}_0$ of the special arc Φ_0 is an arc on Y with order of contact one with E at p .

Let



be the diagram given in Proposition 4.1. Since the wedge Φ does not lift to Y , g' is not an isomorphism, and therefore $\text{Ex}(g') \neq \emptyset$. Let G be the irreducible component of $\text{Ex}(g')$ intersecting the proper transform T of the t -axis ($s = 0$) $\subset S$.

Lemma 5.1. *Every irreducible component of $\text{Ex}(g')$ containing $h(G)$ dominates E .*

Proof. By construction, $p \in \phi(G) \subset E$. Recall that p has codimension one in E , and is not contained in any other irreducible component of $\text{Ex}(f)$. Let C be any irreducible component of $\text{Ex}(g')$ containing $h(G)$. Note that $p \in \phi'(C)$. Since C is irreducible and is contracted by $f \circ \phi'$, and p is a very general point of E , we have $\phi'(C) \subset E$. On the other hand, Lemma 4.2 implies that $\phi'(C) \neq p$. Therefore, as p has codimension one in E , C must dominate E . \square

Lemma 5.2. *The morphism ϕ is dominant.*

Proof. Let $\overline{\phi(Z)} \subset Y$ be the Zariski closure of $\phi(Z)$. Note that $\overline{\phi(Z)}$ is irreducible, since Z is irreducible. By Lemma 5.1, we see that $E \subset \overline{\phi'(Z')} = \overline{\phi(Z)}$. On the other hand, since the lift $\tilde{\Phi}_0$ of the special arc has finite order of contact with E at p , we have $\overline{\phi(Z)} \not\subset E$. As E has codimension one in Y , we conclude that $\overline{\phi(Z)} = Y$. \square

Recall that, by construction, K has transcendence degree $n - 2$ over k . Then we are in the setting of §4.3, and so the relative canonical divisors $K_{Z/Y}$ and $K_{Z'/Y}$ are defined.

The completion of the proof will result from a comparison between the coefficient of G in the relative canonical divisor $K_{Z/S}$, and its coefficient in ϕ^*E . The relative canonical divisors of ϕ and ϕ' will be used to link these two coefficients.

We start from $K_{Z/S}$. Since S is smooth and G is g -exceptional, we have

$$(5.1) \quad \text{ord}_G(K_{Z/S}) \geq 1.$$

Recall that $K_{Z/S} = K_{Z/Z'} + h^*K_{Z'/S}$. Since $K_{Z/Z'} \leq 0$ by Proposition 4.12, we have

$$(5.2) \quad K_{Z/S} \leq h^*K_{Z'/S}.$$

By Proposition 4.9 (see also Corollary 4.11) and the fact that $K_S \sim 0$, we have

$$\phi'^*K_Y \sim K_{Z'} - K_{Z'/Y} \sim K_{Z'/S} - K_{Z'/Y}.$$

We decompose $K_{Z'/Y} = K_{Z'/Y}^{\text{exc}} + K_{Z'/Y}^{\text{hor}}$ where every component of $K_{Z'/Y}^{\text{exc}}$ is g' -exceptional, and none of the components of $K_{Z'/Y}^{\text{hor}}$ is. As Y has terminal singularities, we have $K_{Z/Y} \geq 0$ by Proposition 4.9, and since $K_{Z'/Y} = h_*K_{Z/Y}$, this implies that $K_{Z'/Y}^{\text{hor}} \geq 0$. Therefore $K_{Z'/Y}^{\text{hor}}$, being effective and not containing any g' -exceptional curve, is g' -nef. Note that ϕ'^*K_Y is also g' -nef, because we are assuming that K_Y is f -nef. It follows that $\phi'^*K_Y + K_{Z'/Y}^{\text{hor}}$ is g' -nef. Notice that

$$\phi'^*K_Y + K_{Z'/Y}^{\text{hor}} \sim K_{Z'/S} - K_{Z'/Y}^{\text{exc}},$$

and the \mathbb{Q} -divisor on the right hand side is g' -exceptional. Then Lemma 4.3 implies that

$$(5.3) \quad K_{Z'/S} \leq K_{Z'/Y}^{\text{exc}}.$$

Since g' is the blow-up of an ideal cosupported at the closed point of S , the fiber of g' over this point has pure dimension one, and its support is equal to $\text{Ex}(g')$. As G lies over the closed point of S , it follows that $h(G) \in \text{Ex}(g')$. Let C be any irreducible component of $\text{Ex}(g')$ containing $h(G)$. By Lemma 5.1, C dominates E . Then, by Proposition 4.10, we have

$$\text{ord}_C(K_{Z'/Y}) = \text{ord}_C(\phi'^*E) - 1$$

(this computation can be carried out on Z). Note that there exists at least one such component C because g' is not an isomorphism (if G is not h -exceptional, then $C = h(G)$). Since $K_{Z'/Y}^{\text{exc}}$ is supported on $\text{Ex}(g')$ and ϕ'^*E is effective, the previous formula implies the divisor $\phi'^*E - K_{Z'/Y}^{\text{exc}}$ is effective and nontrivial in a neighborhood of the generic point of $h(G)$. Since Z' is \mathbb{Q} -factorial, it follows that $\phi'^*E - h^*K_{Z'/Y}^{\text{exc}} = h^*(\phi'^*E - K_{Z'/Y}^{\text{exc}})$ is an effective and nontrivial \mathbb{Q} -divisor in a neighborhood of the generic point of G . This means that

$$(5.4) \quad \text{ord}_G(h^*K_{Z'/Y}^{\text{exc}}) < \text{ord}_G(\phi'^*E).$$

Finally, notice that the special arc $\tilde{\Phi}_0: \text{Spec } K[[t]] \rightarrow Y$ factors through a morphism $\psi: \text{Spec } K[[t]] \rightarrow Z$ which gives a parameterization of T . Recall that $\tilde{\Phi}_0$ has order of contact one with E . This means that $\text{ord}_t(\tilde{\Phi}_0^*E) = 1$. Since $\text{ord}_t(\tilde{\Phi}_0^*E) \geq \text{ord}_t(\psi^*G) \cdot \text{ord}_G(\phi'^*E)$, we must have

$$(5.5) \quad \text{ord}_G(\phi'^*E) = 1.$$

Putting (5.1)–(5.5) together, we get

$$1 \leq \text{ord}_G(K_{Z/S}) \leq \text{ord}_G(h^*K_{Z'/S}) \leq \text{ord}_G(h^*K_{Z'/Y}^{\text{exc}}) < \text{ord}_G(\phi'^*E) = 1,$$

which gives the desired contradiction. This completes the proof of Theorem 1.1.

6. TERMINAL VALUATIONS OVER TORIC VARIETIES

Here we give a combinatorial description of terminal valuations in the case of toric varieties.

6.1. Minimal models. In dimension ≥ 3 , minimal models are not unique. Given a variety X , any minimal model program over X , originating from a projective resolution of singularities of X , terminates with a minimal model over X [BCHM10]. Minimal models obtained in this way are \mathbb{Q} -factorial and are all isomorphic in codimension one. Every other minimal model is dominated by a \mathbb{Q} -factorial minimal model via a small birational morphism.

It follows that the set of terminal valuations over X is equal to the set of valuations ord_{E_i} where E_1, \dots, E_m are the prime exceptional divisors on any given minimal model over X . In a way, minimal models play the role of minimal resolutions of surfaces.

6.2. Toric varieties. With regards to toric varieties, we use the notation and terminology of [Ful93]. We fix an algebraic torus T , and denote by M the character lattice of T . We let $N = \text{Hom}(M, \mathbb{Z})$ be the dual lattice, and denote $M_{\mathbb{R}} = M \otimes \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes \mathbb{R}$. The toric variety corresponding to a fan Δ in N is denoted by $X(\Delta)$. We denote by $\Delta(i)$ the set of i -dimensional cones in Δ , and by $\Delta(1)_{\text{prim}}$ the set of primitive elements in the intersection of the rays of Δ with the lattice N . Note that the elements in $\Delta(1)_{\text{prim}}$ correspond to the prime T -divisors on $X(\Delta)$. If $\sigma \subset N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone, then we define, in a similar fashion, $X(\sigma)$, $\sigma(i)$, and $\sigma(1)_{\text{prim}}$ by identifying, in the notation, σ with the fan defined by its faces.

We fix a strongly convex rational polyhedral cone σ , and consider the toric variety $X(\sigma)$. The elements of $\sigma \cap N$ are in bijection with the torus-invariant valuations on $X(\sigma)$. As we shall see below, terminal valuations, Nash valuations, and essential valuations over $X(\sigma)$ are all torus-invariant, and can be characterized in terms of lattice properties of σ .

A cone τ is said to be regular if the vectors in $\tau(1)_{\text{prim}}$ form part of a basis of N . This is equivalent to $X(\tau)$ being smooth. A cone that is not regular is called singular. We write $\sigma_{\text{sing}} = \bigcup_{\tau} \tau^{\circ}$, where τ ranges among the singular faces of σ , and τ° denotes the relative interior of τ . Then $\sigma_{\text{sing}} \cap N$ parameterizes torus-invariant valuations on $X(\sigma)$ whose center is contained in the singular locus $X(\sigma)_{\text{sing}}$. Terminal valuations, Nash valuations, and essential valuations all belong to $\sigma_{\text{sing}} \cap N$.

Given two vectors $v, v' \in \sigma$, we write $v \leq_{\sigma} v'$ if $v' \in v + \sigma$. It is clear that \leq_{σ} is a partial order in σ . In [IK03], the authors prove that Nash and essential valuations coincide, and characterize them in terms of the order \leq_{σ} . To state their result, let

$$\text{Min}(\sigma) := \{\text{minimal elements of } \sigma_{\text{sing}} \cap N \text{ with respect to } \leq_{\sigma}\}.$$

Theorem 6.1 ([IK03]). *With the above notation,*

$$\text{Min}(\sigma) = \{\text{Nash valuations over } X(\sigma)\} = \{\text{essential valuations over } X(\sigma)\}.$$

We denote by $\Gamma(\sigma) \subset N_{\mathbb{R}}$ the convex hull of the non-zero elements in $\sigma \cap N$, and let $\partial_c \Gamma(\sigma)$ be the union of the compact faces of $\Gamma(\sigma)$. Then let

$$\text{Ter}(\sigma) := \partial_c \Gamma(\sigma) \cap \sigma_{\text{sing}} \cap N.$$

Proposition 6.2. *With the above notation,*

$$\text{Ter}(\sigma) = \{\text{terminal valuations over } X(\sigma)\}.$$

Proof. Let Δ be a simplicial subdivision of σ constructed as follows. For every maximal dimensional face F of $\partial_c \Gamma(\sigma)$, we choose a triangulation with set of vertices equal to $F \cap N$. Then we take Δ to be the cone over the resulting triangulation of $\partial_c \Gamma(\sigma)$. Since Δ is simplicial, the corresponding toric variety $X(\Delta)$ is \mathbb{Q} -factorial.

Note that

$$\Delta(1)_{\text{prim}} = \partial_c \Gamma(\sigma) \cap N.$$

Let v_1, \dots, v_m be the elements in this set. Each vector v_i corresponds to a prime T -divisor D_i on $X(\Delta)$, and $X(\Delta)$ has canonical class $K_{X(\Delta)} \sim -\sum_i D_i$. If $n = \dim X(\sigma)$, then the elements in $\Delta(n-1) \setminus \sigma(n-1)$ are in one-to-one correspondence with toric invariant curves that are exceptional over $X(\sigma)$. These curves generate the space of numerical classes of 1-cycles of $X(\Delta)$ over $X(\sigma)$. By the convexity of $\Gamma(\sigma)$, it follows by a standard computation that $K_{X(\Delta)} \cdot \gamma \geq 0$ for any such curve γ (e.g., see the first exercise discussed in [Ful93, Page 99]). This implies that $X(\Delta)$ is a minimal model over $X(\sigma)$.

Since $X(\Delta)$ is a minimal model, the set of terminal valuations over $X(\sigma)$ coincides with the set of valuations defined by the prime exceptional divisors of $X(\Delta)$ lying over the singular locus of $X(\Delta)$. These exceptional divisors are parametrized by the elements in $\Delta(1)_{\text{prim}} \cap \sigma_{\text{sing}}$. By construction, this set is equal to $\text{Ter}(\sigma)$. \square

It is clear from the toric description that $\text{Ter}(\sigma) \subset \text{Min}(\sigma)$. As the following example shows, these two sets can be different.

Example 6.3. Let σ be the cone in \mathbb{R}^3 spanned by the vectors $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $v_3 = (1, 1, 2)$. Then $v = (1, 1, 1) \in \text{Min}(\sigma) \setminus \text{Ter}(\sigma)$, and thus it gives a Nash valuation on $X(\sigma)$ which is not a terminal valuation. Notice, in fact, that $X(\sigma)$ has terminal singularities, and so it has no terminal valuations.

6.3. Minimal valuations. Given any variety X , consider the partial order \leq_X among divisorial valuations on X given by declaring $\nu \leq_X \nu'$ whenever, denoting by $p \in X$ the center of ν' , one has $0 \leq \nu(h) \leq \nu'(h)$ for all $h \in \mathcal{O}_{X,p}$. Note that \leq_X is the same as the order \leq_σ defined above when $X = X(\sigma)$. The order \leq_X depends on the model X . We say that a divisorial valuation ν on X is *minimal* over X if it is a minimal element, among divisorial valuations centered in the singular locus of X , with respect to the partial order \leq_X .

A semi-continuity property of arc valuations implies that every minimal valuation over X is a Nash valuation. On toric varieties, minimal valuations account for all Nash valuations, but this is not true in general. For instance, there is a unique minimal valuation over an E_8 singularity, but there are eight Nash valuations over this singularity.

In a more general setting, every divisorial valuation ν on a variety X defines in a natural way a set $C_X(\nu)$ in the arc space X_∞ , called the *maximal divisorial set* of ν (for instance, the set N_E considered in §1.1 is the same as the maximal divisorial set of the valuation ord_E). The problem of comparing the order \leq_X to the order given by inclusions between maximal divisorial sets has been studied by Ishii in [Ish08], where it is shown that the two orders are different even when $X = \mathbb{A}^n$.

Toric varieties and surfaces form two large classes of varieties for which the Nash map is known to be surjective, but this property seems to hold for different reasons: while in the first case Nash valuations are all minimal, but not necessarily terminal, just the opposite happens in the case of surfaces. This seems to suggest that minimal and terminal valuations complement each other, in some way. We do not know any example of a Nash valuation over a variety X that is neither terminal nor minimal over X .

APPENDIX A. THE CURVE SELECTION LEMMA

Let X be a variety over k . A closed, irreducible subset $N \subset X_\infty$ is said to be *generically stable* if there is an affine open set $U \subset X_\infty$ such that $N \cap U$ is not empty and is cut out, set theoretically, by finitely many equations (i.e., its ideal in $\mathcal{O}_{X_\infty}(U)$ is the radical of a finitely generated ideal). An arc $\alpha \in X_\infty$ is *stable* if its closure is a generically stable set. There

are examples of generically stable sets in X_∞ that are fully contained in $(X_{\text{sing}})_\infty$. In the following, we will focus on generically stable sets that are not contained in $(X_{\text{sing}})_\infty$.

If E is a prime divisor on a resolution of singularities $f: Y \rightarrow X$, then the subset $N_E \subset X_\infty$ given by the closure of $f_\infty(\pi_Y^{-1}(E))$ is a generically stable set and is not contained in $(X_{\text{sing}})_\infty$. In particular, this applies to the irreducible components of $\pi_X^{-1}(X_{\text{sing}})$.

If $\alpha \in X_\infty$ is a stable point that is not contained in $(X_{\text{sing}})_\infty$, then the completion $\widehat{\mathcal{O}_{X_\infty, \alpha}}$ of the local ring of X_∞ at α is Noetherian (cf. [Reg06, Corollary 4.6]). Using this fact, Reguera proved the following form of the curve selection lemma for generically stable sets in arc spaces.

Theorem A.1 ([Reg06, Corollary 4.8]). *Let $N \subsetneq M$ be the inclusion of two irreducible closed sets in X_∞ , and assume that N is generically stable and is not contained in $(X_{\text{sing}})_\infty$. Let $\alpha \in N$ be the generic point, and let κ_α be its residue field. Then there exist a finite algebraic extension L/κ_α , and a morphism $\Psi: \text{Spec } L[[s]] \rightarrow X_\infty$, such that $\Psi(0) = \alpha$ and $\Psi(\eta) \in M \setminus N$.*

Remark A.2. The curve selection lemma can be used to give a slick proof of the Nash problem in dimension one. Suppose indeed that X is a curve with a singularity $p \in X$, and let $f: Y \rightarrow X$ be the normalization. If the Nash problem were false, then we could find a wedge $\Psi: \text{Spec } L[[s, t]] \rightarrow X$ such that the lifts of the special arc Ψ_0 and the generic arc Ψ_η are based at two distinct points in the fiber over p . This is however impossible, since $\text{Spec } L[[s, t]]$ is connected. (There are also more elementary arguments to show this.)

In order to prove Theorem 1.1, we need to take a suitable specialization of the wedge produced by Theorem A.1. This can be achieved using the results of [LJR12, §2.2] (see also [FdB12, §3.2]), but for the convenience of the reader, we include a proof.

We start with some remarks about very general points on arc spaces. Recall that a very general point on a scheme is a point that avoids countably many closed subsets. The existence of such points is clear for schemes of finite type over uncountable fields, but requires some discussion in the case of arc spaces.

Lemma A.3. *Let E be a prime divisor on a resolution of singularities $f: Y \rightarrow X$, and let p be a very general κ_p -valued point on E . Then there exists a very general κ_p -valued point β on N_E . Moreover, β can be chosen in such a way that its lift $\tilde{\beta}$ to Y_∞ verifies $\tilde{\beta}(0) = p$ and $\text{ord}_{\tilde{\beta}}(E) = 1$.*

Proof. Let $\{H_i\}_{i \in \mathbb{N}}$ be a countable collection of hypersurfaces on X_∞ such that no H_i contains N_E , and consider $\tilde{H}_i = f_\infty^{-1}(H_i)$. We will denote by $\pi_m: Y_m \rightarrow Y$, $\tau_m: Y_\infty \rightarrow Y_m$ and $\tau_{m', m}: Y_{m'} \rightarrow Y_m$ the truncation maps. We let $V_{i, m} \subset \pi_m^{-1}(E)$ be the largest subset for which $\tau_m^{-1}(V_{i, m}) \subset \tilde{H}_i$. In other words, $\tau_m^{-1}(V_{i, m})$ is the union of all the fibers of τ_m which are contained in $\tilde{H}_i \cap \pi_Y^{-1}(E)$. Notice that there exists an integer m_0 (depending on i) such that for all $m \geq m_0$, the sets $V_{i, m}$ are hypersurfaces in $\pi_m^{-1}(E)$, and $\tilde{H}_i \cap \pi_Y^{-1}(E) = \tau_m^{-1}(V_{i, m})$. For $m < m_0$, since the truncation map $\tau_{m_0, m}$ has irreducible fibers of dimension $n(m_0 - m)$, where $n = \dim Y$, we see that

$$V_{i, m} = \{\gamma \in \tau_{m_0, m}(V_{i, m_0}) \mid \dim(\tau_{m_0, m}^{-1}(\gamma)) \geq n(m_0 - m)\}.$$

In particular, $V_{i, m}$ is constructible and its closure is a proper subset of $\pi_m^{-1}(E)$. In fact, since $\tau_{m_0, m}^{-1}(V_{i, m})$ is closed [EGA IV₃, Theorem 13.1.3] and $\tau_{m_0, m}$ is smooth, we see that $V_{i, m}$ is closed.

If necessary, we add to the collection $\{\tilde{H}_i\}_{i \in \mathbb{N}}$ one more hypersurface \tilde{H}_0 , in such a way that \tilde{H}_0 contains all arcs $\tilde{\beta} \in Y_\infty$ with $\text{ord}_{\tilde{\beta}}(E) \geq 2$, but does not contain $\pi_Y^{-1}(E)$. Then no

\tilde{H}_i contains $\pi_Y^{-1}(E)$, and therefore each $V_{i,m}$ is a proper subset of $\pi_m^{-1}(E)$. Since the base field k is assumed to be uncountable, we see that there exists a point $\tilde{\beta}_m$ of $\pi_m^{-1}(E)$ not contained in any of the $V_{i,m}$. Notice that $p = \tilde{\beta}_0$ is a very general point of $E = \pi_0^{-1}(E)$.

The result follows if we show that the $\tilde{\beta}_m$ can be chosen compatibly, i.e., in such a way that $\tau_{m+1,m}(\tilde{\beta}_{m+1}) = \tilde{\beta}_m$. To see this, notice that the fiber $\tau_{m+1,m}^{-1}(\tilde{\beta}_m)$ is isomorphic to an affine space \mathbb{A}_L^n , where $n = \dim Y$ and L is the residue field of $\tilde{\beta}_m$. We have that $\tau_{m+1,m}^{-1}(\tilde{\beta}_m) \not\subset V_{i,m+1}$, since $\tilde{\beta}_m \notin V_{i,m}$. Since L is uncountable, we can find an L -valued point $\tilde{\beta}_{m+1} \in \tau_{m+1,m}^{-1}(\tilde{\beta}_m) \simeq \mathbb{A}_L^n$ not contained in any of the $V_{i,m+1}$, and the result follows. \square

Let N be an integral scheme with field of fractions L , and consider a wedge

$$\Psi: \operatorname{Spec} L[[s, t]] \rightarrow X.$$

Given a point $\xi \in N$, we say that Ψ is *defined at ξ* if Ψ factors through $\operatorname{Spec} \mathcal{O}_{N,\xi}[[s, t]]$.

Lemma A.4. *If ξ is a very general point of N , then Ψ is defined at ξ , and hence it induces a wedge $\Phi: \operatorname{Spec} \kappa_\xi[[s, t]] \rightarrow X$ by the diagram*

$$\begin{array}{ccccc} & & \Psi & & \\ & \nearrow & & \searrow & \\ \operatorname{Spec} L[[s, t]] & \longrightarrow & \operatorname{Spec} \mathcal{O}_{N,\xi}[[s, t]] & \longrightarrow & X \\ & & \uparrow & \nearrow \Phi & \\ & & \operatorname{Spec} \kappa_\xi[[s, t]] & & \end{array}$$

We say that Φ is the restriction of Ψ to ξ .

Proof. We need to show that Ψ factors through $\operatorname{Spec} \mathcal{O}_{N,\xi}[[s, t]]$ for a very general $\xi \in N$, the second assertion being a formal consequence of this.

Pick an affine chart $W \subset X$ containing the point $\Psi(0, 0) \in X$, so that Ψ factors through W . Fix an embedding $W \subset \mathbb{A}^d$, and let x_1, \dots, x_d be coordinates in \mathbb{A}^d . Then Ψ is given by equations

$$x_\ell(s, t) = \sum_{i,j} x_{i,j,\ell} s^i t^j \in L[[s, t]].$$

The result follows from the fact that $x_{i,j,\ell} \in \mathcal{O}_{N,\xi}$ for a very general point $\xi \in N$. To see this, let N° be an affine chart of N , of the form $N^\circ = \operatorname{Spec} R$ for some ring R with field of fractions L . Then we can write $x_{i,j,\ell} = a_{i,j,\ell}/b_{i,j,\ell}$, where $a_{i,j,\ell}, b_{i,j,\ell} \in R$. We let $H_{i,j,\ell} \subset N$ be the closure of the hypersurface in N° determined by $b_{i,j,\ell}$, and consider the countable collection of closed sets $\mathcal{H} = \{N \setminus N^\circ\} \cup \{H_{i,j,\ell} \mid i, j, \ell\}$. Then $x_{i,j,\ell} \in \mathcal{O}_{N,\xi}$ for all i, j, ℓ precisely when ξ does not belong to any of the closed sets in \mathcal{H} . \square

We consider now a projective birational morphism $f: Y \rightarrow X$. We assume that the special arc Ψ_0 is not fully contained in the exceptional locus of f .

Lemma A.5. *With the same notation as in Lemma A.4, if ξ is a very general point of N , then Φ lifts to Y if and only if Ψ does.*

Proof. We pick an affine chart $U \subset Y$ containing the point $\tilde{\Psi}_0(0)$. For a very general choice of ξ , the point $\tilde{\Phi}_0(0)$ will also be in U (this can be seen by the same arguments as in the proof of Lemma A.4). We fix a closed embedding $U \subset \mathbb{A}^d$, and let y_1, \dots, y_d be coordinates in \mathbb{A}^d .

In order to describe how Ψ specializes to Φ in these coordinates, it is convenient to interchange the roles of s and t , and consider the wedge $\Psi^\vee: \text{Spec } L[[s, t]] \rightarrow X$ given by

$$\Psi^\vee(s, t) := \Psi(t, s).$$

Switching s and t in the diagram in Lemma A.4 shows that the restriction of Ψ^\vee to ξ gives the wedge $\Phi^\vee: \text{Spec } \kappa_\xi[[s, t]] \rightarrow X$ defined by $\Phi^\vee(s, t) = \Phi(t, s)$. Geometrically, Ψ and Ψ^\vee define isomorphic morphisms to X , and so do Φ and Φ^\vee . In particular, in order to prove the lemma it is equivalent to show that, for a general choice of ξ , Φ^\vee lifts to Y if and only if Ψ^\vee does.

The reason of looking at Ψ^\vee , rather than Ψ , is that if $\tilde{\Psi}_\eta^\vee$ is the lift of the generic arc Ψ_η^\vee of Ψ^\vee , then $\tilde{\Psi}_0(0)$ is a specialization of $\tilde{\Psi}_\eta^\vee(0)$, and therefore $\tilde{\Psi}_\eta^\vee$ factors through U . Similarly, $\tilde{\Phi}_\eta^\vee$ factors through U , and we have the diagram

$$\begin{array}{ccccc} & & \tilde{\Psi}_\eta^\vee & & \\ & \nearrow & & \searrow & \\ \text{Spec } L((s)) & \longrightarrow & \text{Spec } \mathcal{O}_{N, \xi}((s)) & \longrightarrow & U_\infty \\ & & \uparrow & \nearrow \tilde{\Phi}_\eta^\vee & \\ & & \text{Spec } \kappa_\xi((s)) & & \end{array}$$

If $N^\circ = \text{Spec } R$ is an affine chart in N , then the lifts $\tilde{\Psi}_\eta^\vee$ and $\tilde{\Phi}_\eta^\vee$ are given by equations

$$y_\ell(s, t) = \sum_{i, j} \frac{a_{i, j, \ell}}{b_{i, j, \ell}} s^i t^j \in L((s))[[t]], \quad \text{and} \quad y_\ell(s, t) = \sum_{i, j} \frac{\bar{a}_{i, j, \ell}}{\bar{b}_{i, j, \ell}} s^i t^j \in \kappa_\xi((s))[[t]],$$

where $a_{i, j, \ell}, b_{i, j, \ell} \in R$, and $\bar{a}_{i, j, \ell}$ and $\bar{b}_{i, j, \ell}$ are the images of $a_{i, j, \ell}$ and $b_{i, j, \ell}$ in κ_ξ .

Note that if either one of Ψ^\vee and Φ^\vee lifts to Y , then its lift factors through U . Then Lemma 3.2 implies that Ψ (resp., Φ) lifts to Y if and only if $a_{i, j, \ell} = 0$ for all $i < 0$ (resp., $\bar{a}_{i, j, \ell} = 0$ for all $i < 0$). In particular, if Ψ lifts, so does Φ . Conversely, if Ψ does not lift, we have some non-zero element $a_{i, j, \ell}$ with $i < 0$ which defines a hypersurface in N° , and, for any ξ not belonging to this hypersurface, the corresponding Φ does not lift. \square

Theorem A.6. *Let $f: Y \rightarrow X$ be a projective birational morphism of varieties, and let $\alpha \in X_\infty$ be an arc that is not fully contained in $(X_{\text{sing}})_\infty$. Suppose that*

$$\Psi: \text{Spec } L[[s]] \rightarrow X_\infty$$

is a morphism such that L/κ_α is a finite algebraic extension of the residue field of α , and $\Psi(0) = \alpha$. Let $N \subset X_\infty$ be the closure of α , and let β be a very general point of N . Then there is a finite algebraic extension K/κ_β , and a morphism

$$\Phi: \text{Spec } K[[s]] \rightarrow X_\infty,$$

such that $\Phi(0) = \beta$. Moreover, Φ factors through Y_∞ if and only if Ψ does.

Proof. We consider generators ζ_1, \dots, ζ_l for the field extension L/κ_α . We pick an affine chart $N^\circ = \text{Spec } R \subset N$, where R is a domain with field of fractions κ_α , and ζ_1, \dots, ζ_l are integral over R . If R' is the R -algebra generated by ζ_1, \dots, ζ_l , and we consider $N' = \text{Spec } R'$, then the natural map $\rho: N' \rightarrow N^\circ$ is finite. Notice that L is the field of fractions of R' .

Since R' is integral over R , the image $\rho(H)$ of a proper closed subset $H \subsetneq N'$ is a proper closed subset of N° . Therefore, the image of a very general point in N' is a very general point of N , and a very general point in N is the image of very general point in N' . For our very

general point $\beta \in N$, we let β' denote a very general point in N' such that $\rho(\beta') = \beta$. We set $K = \kappa_{\beta'}$. Since the map ρ is finite, we see that K is a finite extension of κ_{β} .

By Lemma A.4, the wedge Ψ is defined at β' , and restricts to a wedge $\Phi: \text{Spec } K[[s]] \rightarrow X_{\infty}$. By construction, it is clear that $\Phi(0) = \beta$. Moreover, by Lemma A.5, the wedge Ψ lifts to Y if and only if Ψ does, as required. \square

Proof of Theorem 3.3. The assumption of the theorem means that the closure N_E of α is strictly contained in an irreducible component M of $\pi_X^{-1}(X_{\text{sing}})$. Note, on the other hand, that $N_E \not\subset (X_{\text{sing}})_{\infty}$. Then, denoting by κ_{α} the residue field of α , Theorem A.1 implies that there is a finite algebraic field extension L/κ_{α} , and a morphism $\Psi: L[[s]] \rightarrow X_{\infty}$, such that $\Psi(0) = \alpha$ and $\Psi(\eta) \in M \setminus N_E$. Note, in particular, that Ψ does not lift to Y , since $\tilde{\alpha}$ is the generic point of an irreducible component of $\pi_Y^{-1}(f_{\infty}^{-1}(X_{\text{sing}}))$. By Lemma A.3, we can find a very general κ_p -valued point $\beta \in N_E$ that lifts to a κ_p -valued point $\tilde{\beta} \in Y_{\infty}$ such that $\tilde{\beta}(0) = p$ and $\text{ord}_{\tilde{\beta}}(E) = 1$. Then the assertion follows from Theorem A.6. \square

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